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Three Limit Cycles for a Three-Dimensional Lotka-Volterra Competitive System with a Heteroclinic Cycle

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Abstract—Three limit cycles are constructed for a three-dimensional Lotka-Volterra competitive system with a heteroclinic cycle. This gives a partial answer to a problem proposed by Hofbauer and So in [1].

Keywords—Lotka-Volterra systems, Competitive, Hopf bifurcation, Limit cycles.

1. INTRODUCTION

Consider the following three-dimensional Lotka-Volterra competitive system:

$$\dot{x} = \operatorname{diag}(x)A(x-1), \tag{1}$$

where $x = (x_1, x_2, x_3)$, $A = (a_{ij})_{3\times 3}$ with $a_{ij} < 0$ $(i, j = 1, 2, 3)$, $x - 1 = (x_1 - 1, x_2 - 1, x_3 - 1)$.

Set $\bar{x} = x - 1$. Then system (1) takes the form

$$\dot{x} = \operatorname{diag}(1 + x)Ax. \tag{2}$$

Here we used x_i instead of $\bar{x}_i$ for $i = 1, 2, 3$.

Based on Hirsch's monotone theory [2], it is proved by Zeeman [3] that the ω-limit set of a three-dimensional Lotka-Volterra competitive system belongs to a two-dimensional carrying simplex. Zeeman uses geometric analysis of the surfaces $\dot{x}_i = 0$ of system (1) and gives a classification of 33 stable equivalence classes for these systems in [3]. It is shown that the dynamical behaviors

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for the systems have been fully described in 27 of these classes, for all the compact limit sets are fixed points [3,4]. The Hopf bifurcation theorem is applied to show that the remaining classes 26, 27, 28, 29, 30, and 31 can possess isolated periodic orbits (limit cycles). The question of how many limit cycles can appear in Zeeman's six classes 26–31 remains open.

Hofbauer and So [1] and Xiao and Li [5] construct systems with two limit cycles in class 27. In their cases, the local stable positive equilibrium is surrounded by two limit cycles, in which one is from the Hopf bifurcation theorem and the other is guaranteed by the Poincaré-Bendixson theorem. Lu and Luo [6] construct systems of classes 26, 27, 28, and 29 with two small amplitude limit cycles which show that competitive systems without heteroclinic cycles (which belong to Zeeman's classes 26, 28, and 29) can also have two limit cycles.

In [1], Hofbauer and So propose a question as to whether there can be three or more limit cycles in three-dimensional Lotka-Volterra competitive systems.

The purpose of this paper is to construct an example to show that a three-dimensional Lotka-Volterra competitive system with a heteroclinic cycle (which belongs to Zeeman's class 27) can have three limit cycles. The construction is similar to that of Hofbauer and So [1]. In our case, more complicated symbolic computation is involved and the procedure is mechanical.

2. THREE LIMIT CYCLES

Consider system (2) with the interaction matrix A as follows:

$$A = \begin{pmatrix} -1 & -2 & \lambda \\ -\frac{3}{5} & -3 & \mu \\ -2 & -\frac{11}{10} & -5 \end{pmatrix}. \tag{3}$$

To prove the existence of three limit cycles in system (2), we choose A such that the origin has a Hopf bifurcation. Suppose A has one negative real eigenvalue λ and a pair of purely imaginary eigenvalues $\pm\omega i$ ($\omega \neq 0$). To satisfy these necessary eigenvalue conditions, we need $\det(A) = (A_{11} + A_{22} + A_{33})\,\mathrm{tr}(A)$; i.e., $\mu = -117/8 - (633/640)\lambda$.

In [6], it is shown that one can construct a matrix T whose elements are rational functions of the elements of matrix A such that

$$TAT^{-1} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{4}$$

Here, the submatrix

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

has a pair of purely imaginary eigenvalues $\pm\omega i$ ($\omega \neq 0$). That is, $c_{11} + c_{22} = 0$ and $c_{11}c_{22} - c_{12}c_{21} > 0$.

For system (2) with matrix (3), we can get the transformation matrix T as follows:

$$T = \begin{pmatrix} -\frac{3}{5} & 6 & -\frac{117}{8} - \frac{633}{640}\lambda \\ -2 & -\frac{11}{10} & 4 \\ \frac{633}{80} - \frac{6963}{6400}\lambda & 8 - \frac{11}{10}\lambda & \frac{117}{4} - \frac{1287}{320}\lambda \end{pmatrix}.$$

Using the transformation $y = Tx$, we get a new system from system (2),

$$\dot{y} = T * \mathrm{diag}\left(T^{-1}y + \mathbf{1}\right) AT^{-1}y = Cy + T * \mathrm{diag}\left(T^{-1}y\right) AT^{-1}y, \tag{5}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{1} = (1,1,1)$, and $C = TAT^{-1}$ is a block diagonal matrix.

The linear part Cy of system (5) takes the form

$$Cy = \begin{pmatrix} -\frac{505}{211} & -\frac{20943}{1688} - \frac{633}{640}\lambda & 0 \\ \frac{5837}{6330} & \frac{505}{211} & 0 \\ 0 & 0 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

From the center manifold theorem [7], we can suppose that the transformed system (5) with linear part Cy has an approximation to the center manifold which takes the form

$$y_3 = h(y_1, y_2) = f_2(y_1, y_2) + f_3(y_1, y_2) + f_4(y_1, y_2) + h.o.t, \tag{6}$$

where $f_i = \sum_{j=0}^{i} d_{ij} * y_1^{i-j} * y_2^j$ and $h.o.t$ denotes the terms with degrees greater than or equal to five.

Substituting $y_3 = h(y_1, y_2)$ into both sides of the third equation of $\dot{y}$ in system (5), we have

$$\dot{y}_3 = \text{subs}\left(y_3 = h(y_1, y_2), \left(Cy + T * \text{diag}\left(T^{-1}y\right) AT^{-1}y\right)\{3\}\right), \tag{7}$$

and also

$$\dot{y}_3 = \dot{y}_1 * \frac{\partial h}{\partial y_1} + \dot{y}_2 * \frac{\partial h}{\partial y_2}. \tag{8}$$

Here $\{3\}$ at the end of equation (7) denotes the third element of the vector.

Comparing the coefficients of the former two equations, we can get the coefficients d_{ij} of f_i as the rational function of λ. Theoretically, we can get an arbitrary-order approximation to the center manifold.

To compute the focal values up to order two on the center manifold, we need to calculate the approximate center manifold $y_3 = h(y_1, y_2)$ up to f_4. The calculated h is a polynomial of y_1, y_2 with 12 terms whose coefficients are rational functions of λ.

Hence,

$$y_3 = h(y_1, y_2) + h.o.t = \frac{\text{numer}}{\text{denom}} + h.o.t,$$

where

$$\begin{aligned} \text{numer} :=& -5\left(e_{20}y_1^2 + e_{21}y_1y_2 + e_{22}y_2^2 + e_{30}y_1^3 + e_{31}y_1^2y_2 + e_{32}y_1y_2^2 + e_{33}y_2^3 + e_{40}y_1^4\right. \\ &\left. + e_{41}y_1^3y_2 + e_{42}y_1^2y_2^2 + e_{43}y_1y_2^3 + e_{44}y_2^4\right), \\ \text{denom} :=& 1284413397768(5837\lambda + 94160)(5837\lambda + 166160)^3(5837\lambda + 554960)^6 \\ &\times (5837\lambda + 68960). \end{aligned}$$

The coefficients e_{ij} of numer are given in the Appendix.

Substituting $y_3 = h(y_1, y_2)$ into the equations of $\dot{y}_1$ and $\dot{y}_2$, we obtain a two-dimensional system of center-focus type. We will consider the stability and Hopf bifurcation of the equilibrium on the approximate center manifold.

A classic way to deal with the limit cycles of a two-dimensional system with center focus form is to construct its Liapunov function, then calculate and perturb its focal values; see [8]. Under the computer algebraic system Maple, we have written a program Liapc to calculate the focal values which is similar to the program which appeared in [9]. Using our Maple program Liapc, we get the first two focal values

$$LV_1 = f(\lambda) = \frac{f_1(\lambda)}{f_2(\lambda)},$$

$$LV_2 = g(\lambda) = \frac{g_1(\lambda)}{g_2(\lambda)},$$

where

$$
\begin{aligned}
f_1(\lambda) &= -63300(5837\lambda + 36560)\left(194341522781\lambda^3 + 4397246049014\lambda^2\right.\\
&\quad \left.+ 28169289409536\lambda + 43772534807040\right),\\
f_2(\lambda) &= 34070569(5837\lambda + 166160)(5837\lambda + 554960)^2,\\
g_1(\lambda) &= 42200\,(386439789013918447668447903884461628668362836243582017278534495 5\lambda^{13}\\
&\quad - 92845259967555514903323769071813289905894700112535899550715207887 8\lambda^{12}\\
&\quad - 1283725157987640624968235864561013890405611602814012673972613672352 64\lambda^{11}\\
&\quad - 5825846765270495342142961809928056310096210217677428414085639883499520\lambda^{10}\\
&\quad - 1155485191227921780312220072729843718121036108458710716657620708184064 00\lambda^{9}\\
&\quad - 5119879609317002666808041798963845588187759545165294122886544690872320 00\lambda^{8}\\
&\quad + 2086012116681170043651935704361045175464995287199045623207651090890752 0000\lambda^{7}\\
&\quad + 4322729918437005493704072004000243750243474971081606945966004665057280 00000\lambda^{6}\\
&\quad + 3603893514009811028511976902737128506162456591099452651593418347118592 000000\lambda^{5}\\
&\quad + 1244733306220548941610823310478093258258607139358281501031470567784448 0000000\lambda^{4}\\
&\quad - 9398967501645520162369529874613256571397140465952165667095749171609600 000000\lambda^{3}\\
&\quad - 2045759908278177699622383408026411421673349100526871126964707776266240 00000000\lambda^{2}\\
&\quad - 6076166983748199991450574412928534129574783754979308620225253369446400 00000000\lambda\\
&\quad - 7514093797160190748069819462415733644329261384468078906029989953536000 0000000),\\
g_2(\lambda) &= 677561103336921295 7(44521\lambda + 558480)^2(5837\lambda + 94160)(5837\lambda + 166160)^3(5837\lambda + 554960)^6.
\end{aligned}
$$

According to the Liapunov theorem and Hopf bifurcation theory, these focal values determine how many limit cycles can be constructed by perturbing the coefficients of the system.

In [10], we have proposed an algorithm `mrealroot` which extends the Maple algorithm `realroot`. This algorithm can isolate the real roots of multiple polynomials and determine the signs of other polynomials at these real roots. Using the `mrealroot` algorithm in [10], we have written a Maple program to deal with the focal values. We run the `mrealroot` command with parameters as

$$
\operatorname{mrealroot}\left([f_1(\lambda)], [\lambda], \frac{1}{10^{20}}, [f_2(\lambda), g_1(\lambda), g_2(\lambda), \det(A), \lambda, \mu]\right),
$$

and get the output

$$
\left[\left[-\frac{337819463133526326181}{147573952589676412928}, -\frac{844548657833381581545}{368934881474191032 32}\right], [+,-,+,-,-,-]\right].
$$

This shows that there is one real root of numer(LV_1) (unknown λ) in the interval

$$
\left[-\frac{337819463133526326181}{147573952589676412928}, -\frac{844548657833381581545}{36893488147419103232}\right]
$$

(with length $1/10^{20}$) which makes denom$(LV_1) > 0$, numer$(LV_2) < 0$, denom$(LV_2) > 0$, $\det(A) < 0$, $\lambda < 0$, $\mu < 0$.

$\lambda < 0$ and $\mu < 0$ mean that A is a competitive system. $\det(A) < 0$ shows that A satisfies the condition of existence of a pair of imaginary eigenvalues, and in this case, the real root of LV_1 makes $LV_2 < 0$.

This means that system (2) with matrix (3) may have two small amplitude limit cycles. To bifurcate two limit cycles, first perturb λ such that $LV_1 LV_2 < 0$ and adjust μ such that $\mu = -117/8 - (633/640)\lambda$ which keeps the linear part of the system in a center-focus form. One limit cycle bifurcates. For the second limit cycle, perturb μ such that the real parts of the complex roots (which are a pair of conjugate ones) are of the opposite sign to LV_1.

The remaining work is to check to which class in Zeeman's classification the constructed system belongs.

Using Zeeman's notation, we have $R_{ij} = \mathrm{sgn}(\alpha_{ij})$ and $Q_{kk} = \mathrm{sgn}(\beta_{kk})$, with $\alpha_{ij} = b_i a_{ji}/a_{ii} - b_j = (AR_i)_j - b_j$ and $\beta_{kk} = (AQ_k)_k - b_k$, which are the algebraic invariants of A. Here, R_i is the equilibrium on the x_i-axis, and Q_k is the positive equilibrium on the plane of $x_k = 0$.

If $R_{12} = R_{23} = R_{31}$, $R_{21} = R_{13} = R_{32}$, and $R_{ij}R_{ji} < 0$, then the system has a heteroclinic cycle which is a repellor (or attractor) when $p = -(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{21}\alpha_{13}\alpha_{32}) > 0$ (or < 0). This is case 27 of Zeeman's classification.

Since

$$R_{12} = 1, \qquad R_{23} = 1, \qquad R_{31} = 1, \qquad R_{13} = -1, \qquad R_{21} = -1, \qquad R_{32} = -1,$$
$$\operatorname{sign}(R_{ij}) = -\operatorname{sign}(R_{ji}),$$

and

$$p = -(\alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{21}\alpha_{13}\alpha_{32}) < 0,$$

this system with two small amplitude limit cycles belongs to class 27 in Zeeman's classification. Since the second focal value is negative, so the outer limit cycle is stable. Furthermore, negative p ensures that the heteroclinic cycle is an attractor. By the Poincaré-Bendixson theorem, we obtain the third limit cycle.

As a result, we have constructed a three-dimensional Lotka-Volterra competitive system with three limit cycles.

3. DISCUSSION

In Section 2, we have shown how to construct the limit cycles of a three-dimensional Lotka-Volterra competitive system. By Hirsch's monotone theory, the ω-limit set of an orbit (except the positive equilibrium) of the three-dimensional Lotka-Volterra competitive system is contained in a two-dimensional carrying simplex. First we transform the original system to the system with the linear part which takes a block-diagonal form. Then we compute the approximate center manifold (up to the $2k^{\text{th}}$-order terms) of the transformed system and get a two-dimensional system with center-focus type on this manifold. Applying the Poincaré-Liapunov method, the Liapunov function and focal values (up to k^{th} order) can be computed. Finally, by calculating the focal values and checking the stability of the heteroclinic cycle, we can get three limit cycles according to bifurcation theory and the Poincaré-Bendixson theorem.

All these steps can be done mechanically. If we give the coefficients matrix A (with symbolic coefficients) of system (2) as the parameter of our program $3DLVL$, we can get the following information from the output: what conditions the system should satisfy to be a competitive one with Hopf bifurcation, how many small amplitude limit cycles the system can have by perturbing the symbolic coefficients, what class in Zeeman's classification the system belongs to, and whether the heteroclinic cycle is an attractor or when the system belongs to class 27. Theoretically, we can consider the interaction matrix A with more (than two) symbolic elements, and calculate the focal values up to order k ($k \geq 3$) to the transformed two-dimensional system on the center manifold. Then we can construct more than two small amplitude limit cycles in this system by perturbing each parameter step by step. In fact, for the computational obstacle of rapid increase of the terms and degrees of the polynomials in the manipulation, it seems impossible to calculate the focal values with orders higher than two if there are no more efficient algorithms.

APPENDIX

$$\begin{aligned}
e_{20} :=& -307729152(11\lambda - 80)(5837\lambda + 94160)(5837\lambda + 68960)\left(9151457045107\lambda^2\right.\\
&\left.- 1875141700485534\lambda + 2186378748334880\right)(5837\lambda + 166160)^2(5837\lambda + 554960)^4,\\
e_{21} :=& \ 9231874560(11\lambda - 80)(5837\lambda + 94160)(5837\lambda + 68960)\left(14227180445647\lambda^2\right.\\
&\left.+ 14380946535250\lambda - 1527317233759200\right)(5837\lambda + 166160)^2(5837\lambda + 554960)^4,
\end{aligned}$$

$$
\begin{aligned}
e_{22} :=\ & 43274412(11\lambda - 80)(5837\lambda + 94160)(5837\lambda + 68960)\big(69801613689431\lambda^3 \\
& - 5549088695637142\lambda^2 - 154773025429264320\lambda - 636498286129804800\big) \\
& \times (5837\lambda + 166160)^2(5837\lambda + 554960)^4, \\
e_{30} :=\ & -829685760(11\lambda - 80)(5837\lambda + 68960)\big(502748296973006392836411139\lambda^4 \\
& - 5811026285782250614319805268\lambda^3 + 120788895033133797084639822720\lambda^2 \\
& + 724278073066951533114986572800\lambda + 1555685610006016961355407360000\big) \\
& \times (5837\lambda + 554960)^2(5837\lambda + 166160)^2, \\
e_{31} :=\ & -9333964800(11\lambda - 80)(5837\lambda + 68960)\big(22117080468309062143022 7\lambda^5 \\
& - 1788518467547051303516048 96\lambda^4 + 443973506073994249368064 2804\lambda^3 \\
& + 890674911144783247102929955 20\lambda^2 + 51582100510019253461264358400\lambda \\
& + 30654113553635626868017152000\big)(5837\lambda + 554960)^2(5837\lambda + 166160)^2, \\
e_{32} :=\ & 1750118400(11\lambda - 80)(5837\lambda + 68960)\big(45913610794354501633903039\lambda^5 \\
& - 611312737925094193626272167 0\lambda^4 - 1381418519300277206074132354 56\lambda^3 \\
& - 1220199203268997772237967718 40\lambda^2 + 7715043057985306542639492300800\lambda \\
& + 10208781928529661238933536768000\big)(5837\lambda + 554960)^2(5837\lambda + 166160)^2, \\
e_{33} :=\ & 1367280(11\lambda - 80)(5837\lambda + 68960)\big(5433375731433488008904301 23\lambda^6 \\
& - 24783790940057549066173856468 6\lambda^5 + 2361276866630151707371735737120\lambda^4 \\
& + 36303798732678465596198806929 9200\lambda^3 + 61427855807636055291496519782 40000\lambda^2 \\
& + 3401246360997122828292972060 6720000\lambda + 4240996628280288781345898496 0000000\big) \\
& \times (5837\lambda + 554960)^2(5837\lambda + 166160)^2, \\
e_{40} :=\ & 102400(11\lambda - 80) \\
& \times \big(82949753156648614772188154074729059920822003601\lambda^9 \\
& - 1999546355933849649550454380839140633835824213165 06\lambda^8 \\
& + 837167907678832231734652813666310265474009772822860 8\lambda^7 \\
& + 2020343921065613700640556843679547961482170532708454 40\lambda^6 \\
& - 2073159130182922622011584498509276471156179849393438 7200\lambda^5 \\
& - 3317349570601907047005118424162498758396715256209408 00000\lambda^4 \\
& + 5274739129724205793274540487816110180500375898475724 800000\lambda^3 \\
& + 7155416425932042994104934651973897791052750817738096 6400000\lambda^2 \\
& + 2236086771119667931339552240053098579675756738662891 52000000\lambda \\
& + 2247185228779081706145475043182493411883321720727142 40000000\big), \\
e_{41} :=\ & -3072000(11\lambda - 80) \\
& \times \big(720999972571209040139439366085705408271041732179\lambda^9 \\
& - 2469499717241182811662763041356035404671492798651 02\lambda^8 \\
& - 236859477347010627702339086385882413016072315287987 2\lambda^7 \\
& + 5588442924449894479364796806482018594082574920358604 80\lambda^6 \\
& + 1121462119973414294340029916611952567453348260657070 0800\lambda^5 \\
& - 1041933064961057827987590538952446683269404032691404 80000\lambda^4 \\
& - 3134199258287288512048721046883741518804653946175488 000000\lambda^3 \\
& - 1510132893815612051290875031521601131820643060453212 1600000\lambda^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 6462722159813792017715066651285311962817618033770496000000\lambda \\
&\quad + 296640721334788173639465331472292525279405695985254400000000\Big), \\
e_{42} &:= -28800(11\lambda - 80) \\
&\quad \times \Big(1570335674045587104267472292569144322617489610787\lambda^{10} \\
&\quad - 22375670187005996018290524446219053376349390885731 18\lambda^{9} \\
&\quad + 23375844490091295114901139555506079305250347958958 1728\lambda^{8} \\
&\quad + 210108326468354730849867842425160196714798058236559 46240\lambda^{7} \\
&\quad + 2931196314456990695569366411881108643886892210778480 64000\lambda^{6} \\
&\quad - 72481313379061622237046669499425469270086258264101683 20000\lambda^{5} \\
&\quad - 2065537094034118977794275517508887102104993948222226432 00000\lambda^{4} \\
&\quad - 14692050060731243605168378971068114630925649259974885376 00000\lambda^{3} \\
&\quad - 811830997342770286190318014851059648585502771042582528 000000\lambda^{2} \\
&\quad + 16810623021793395953612518854011172463253590987558092800 000000\lambda \\
&\quad + 254169753140848920335199364726465520410549502953390080000 00000\Big), \\
e_{43} &:= 432000(11\lambda - 80) \\
&\quad \times \Big(2931802491252491824508456841496159599924323683767\lambda^{10} \\
&\quad - 12645586656428375084017112039941232040386169749621 18\lambda^{9} \\
&\quad - 36438747852118729517751104486154787825068191269808 992\lambda^{8} \\
&\quad + 397904999512149706546385235964247758663574616073540 6080\lambda^{7} \\
&\quad + 2229692242332232835934792114412245849143513327475040 25600\lambda^{6} \\
&\quad + 41777486517532239185799965073586944602875254023139819 52000\lambda^{5} \\
&\quad + 2784675950485565398082507227886472763160315446269050 8800000\lambda^{4} \\
&\quad - 69854532936054144303002643944311578957037395353639321 600000\lambda^{3} \\
&\quad - 165137881042609802858204124730143221954487437250632089 6000000\lambda^{2} \\
&\quad - 5825434614570271029131221103378377575011989038060011520 000000\lambda \\
&\quad - 4935886728733828477097806365621089975257795974817382400 000000\Big), \\
e_{44} &:= 2025(11\lambda - 80) \\
&\quad \times \Big(3334758277933400612753909197856177138956217232 33\lambda^{11} \\
&\quad - 4516911255923970251513105093107673895834944829873434\lambda^{10} \\
&\quad + 17079463760157869773042562209871008134566312180262905 6\lambda^{9} \\
&\quad + 309422046976186357772740418150310401517491155295640038 40\lambda^{8} \\
&\quad + 505550519620834815176942310202433602428306199507007897 600\lambda^{7} \\
&\quad - 3230507798921390273078790428242964874729323844169082470 4000\lambda^{6} \\
&\quad - 1507744050895355923559010131297704737977361293785867223 040000\lambda^{5} \\
&\quad - 2730068882208122861015488940552576417158749306697778462 7200000\lambda^{4} \\
&\quad - 2552505495744670577552738898762443297194375184790511943 68000000\lambda^{3} \\
&\quad - 1244225627136308625999371553402763408215796197640969912 320000000\lambda^{2} \\
&\quad - 2792124616259019941540202211792032764397190688376304435 200000000\lambda \\
&\quad - 2044666461217245318168651559068193567101312292651794432 000000000\Big).
\end{aligned}
$$

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